§ 4. A Few Deductions.

The given theorems may be used to advantage to find the numbers which belong to certain exponents, especially when the ϕ-function of these exponents is small. Since the reciprocal of any number belongs to the same exponent as the number itself, it results that when the ϕ-function of this exponent is 2 the number and its reciprocal constitute the only numbers which belong to this exponent. In particular, when the number n belongs to exponent 3(mod p) n^2 must belong to the same exponent and $n^2+n\equiv -1$ (mod p). Hence we may say that a necessary and sufficient condition that the number $n>1$ belongs to exponent 3 (mod p) is that $n(n+1)\equiv-1$. Hence n belongs to exponent 3 whenever p is of the form $n(n+1)+1$. The five primes below 100 which are of this form are 7, 13, 31, 43 and 73. Numbers belonging to exponent 3 (mod p) may often be readily obtained by the following method, whose correctness is easily proved: Find the reciprocal r of 4 (mod p) and find by trial the smallest value of k such that $kp+r-1$ is a perfect square. The two numbers $\frac{1}{2}(p-1)\pm\sqrt{(kp+r-1)}$ will then belong to exponent 3. It is clear that p must have the form $6n+1$.

In a similar manner we see that if n belongs to exponent 4, n^3 will belong to this exponent and $n^3+n\equiv 0 (\text{mod } p)$. Two neccessary and sufficient conditions that the number n belongs to exponent 4 (mod p) are therefore that $n(n^2+1)\equiv 0$, and that n is prime to p. Similarly, we observe that a necessary and sufficient condition that n belongs to exponent 6 (mod p) is that $n+\dfrac{1}{n}\equiv 1$ (mod p). We may deduce from these results the following useful theorem: *A necessary and sufficient condition that the number n, which is prime to the prime odd number p, belongs to exponents 3, 4, or 6 (mod p) is that the sum of n and its reciprocal is congruent to -1, 0, or 1 respectively.* When this sum is congruent to ± 1, n must be prime to p. Hence we have that a necessary and sufficient condition that n belongs to exponent 3 or 6 (mod p) is that $n+\dfrac{1}{n}\equiv -1$ or $\equiv 1$, respectively.

ON THE REPRESENTATION OF AN INTEGER AS THE SUM OF CONSECUTIVE INTEGERS.

By THOMAS E. MASON, Indiana University.

Lucas has shown that every number not of the form 2^n can be expressed as the sum of two or more consecutive positive integers. In this paper we shall consider series of consecutive integers and shall not exclude zero and negative terms. It is proposed to find the number of ways in

which a number may be expressed as a sum of consecutive integers, including the case of a single term.

The question resolves itself into two problems.

Case 1. *When the number of terms in the series is odd.*

In this case we have

$$\text{(A)} \qquad m=(2n+1)a,$$

where m is the number, $2n+1$ the number of terms in the series, and a the mid-term. For every such factorization of m there exists a series with m as the sum, and for every series there exists the corresponding factorization. Hence the problem reduces to the problem of finding the number of ways in which m can be factored into two factors, one of which is odd. When the number is 2^n there is only the one odd factor 1. In every other case if we express m thus,

$$m=2^a \,.\, p_1^{\alpha_1}.p_2^{\alpha_2}.p_3^{\alpha_3}.p_4^{\alpha_4}.p_5^{\alpha_5} \;.\;.\;.\; p_r^{\alpha_r},$$

where the p's are distinct odd primes and the α's are the powers to which they occur, then by the theorem for the number of factors of an integer, we have

$$(\alpha_1+1)(\alpha_2+1)(\alpha_3+1) \;.\;.\;.\; (\alpha_r+1)$$

as the number of ways in which we can factor m into two factors, one odd.

Case 2. *When the number of terms in the series is even.*

For this case we have

$$\text{(B)} \qquad m=2n(2a+1)/2=n(2a+1),$$

where $2n$ is the number of terms and where a and $a+1$ are the pair of terms in the middle of the series. We see that the problem reduces again to that of finding the number of ways in which m can be separated into two factors, one being odd; the result will evidently be the same as in the first case.

Therefore the total number of ways in which m can be expressed as the sum of consecutive integers is

$$2(\alpha_1+1)(\alpha_2+1)(\alpha_3+1) \;.\;.\;.\; (a_r+1),$$

except in the case where m is 2^n, in which the total number is 2.

Since the factor 2^a has no part in determining the number of ways in which the number may be factored into two factors, one odd, we shall have an infinity of numbers in geometrical series, with ratio two, that can be expressed as the sum of consecutive integers in the same number of ways.

Every factorization of m into two factors, one odd, gives two series. For if $m=(2n+1)a$, we may consider $2n+1$ the number of terms in the series and a the mid-term, or we may take $2a$ as the number of terms and n and $n+1$ as the pair of terms in the middle of the series. Let us consider

$$m=(2n+1)a,$$

where $2n+1$ is the number of terms and a the mid-term. There can be but $a-1$ terms less than a which do not include zero or zero and negative terms. Obviously there will be n terms of the series less than a. Therefore the series will contain all positive terms only when

$$(1) \qquad a-n>0.$$

Let us now consider $2a$ the number of terms of the series and n and $n+1$ the pair of terms in the middle of the series. There can be but $n-1$ terms less than n which do not include zero or zero and negatives. Also there can be but $a-1$ terms of the series below n. Therefore the series will contain only positive terms when

$$(2) \qquad n-a \geqq 0.$$

The relations (1) and (2) are such as to show that when one series contains all positive terms the other will contain a zero or zero and negatives. Since each factorization of m gives one series with all terms positive and one series containing a zero or zero and negatives, we shall have half the total number of series with terms all positive and half with zero or zero and negatives.

From (1) we have that the condition for all positive integers in the series is

$$\text{From (A)}: \quad a>n, \quad \frac{m}{2n+1}=a>n, \quad \frac{m}{2n}>\frac{m}{2n+1}>n.$$

$$\therefore \frac{m}{2}>n^2, \text{ or } n<\sqrt{\frac{m}{2}}, \text{ or } 2n<\sqrt{(2m)}.$$

That is, when one less than the number of terms is less than the

square root of twice the number (m) the series consists of positive integers. From (B), in like manner we get

$$2a < \sqrt{(2m)}$$

as the condition for all positive terms, where $2a$ is the number of terms in the series.

Since m can be expressed as the sum of n numbers, it can also be expressed as the sum of n series of consecutive integers. The number of such sets of series will equal the combinations of n things, one being taken from each of n groups, where the number in the group is found by the method of this paper. As m can also be expressed as a difference, a product, or a quotient, it can be expressed as the difference, the product, or the quotient, of series or sets of series.

We shall gather our results into the following

THEOREM. *If we consider a number itself as a series of consecutive integers of one term, and do not exclude zero and negative terms, then a number*

$$m = 2^{a} \cdot p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot p_3^{\alpha_3} \cdot p_4^{\alpha_4} \ldots p_r^{\alpha_r},$$

where the p's are distinct odd primes and the α's the exponents of the powers to which they appear, can be expressed as the sum of consecutive integers in

$$2(\alpha_1+1)(\alpha_2+1)(\alpha_3+1) \ldots (\alpha_r+1)$$

ways, except in the case $m=2^n$ where the number of ways is 2.

One half the number of series will contain an even number of terms and one half an odd number of terms.

Also one half the number of series will be composed of all positive terms and one half will contain zero or zero and negatives.

An example illustrating the above theorem is the following:

$$15 = 3 \times 5$$

Therefore the number of series will be

$$2(1+1)(1+1) = 2^3 = 8.$$

Number of Terms.	Mid-Term or Pair of Mid-Terms.	The Series.
1	15	15
3	5	$4+5+6$
5	3	$1+2+3+4+5$
15	1	$-6-5-4-3-2-1+0+1+2+3+4+5+6+7+8$
2	7,8	$7+8$
6	2,3	$0+1+2+3+4+5$
10	1,2	$-3-2-1+0+1+2+3+4+5+6$
30	0,1	$-14-13-\ \ldots\ +0+1\ \ldots\ +14+15$

NOTE ON PRIME NUMBERS.

By DERRICK N. LEHMER, University of California.

It is a well known theorem that it is possible to find an arbitrarily great number of consecutive composite numbers. This appears from the values which the expression $n!+r$ takes for $r=2, 3, \ldots, n$. This theorem furnishes an interesting proof of the theorem that the number of primes less than or equal to x is not determined by a function of x which is a polynomial in x of finite degree. For if $f(x)$ were such a function of degree n, then for $x=(n+2)!+r$, $f(x)$ must keep the same value for $r=2, 3, 4, \ldots, n+2$. If this value is k, then $f(x)-k=0$ is an equation of degree n with $n+1$ roots, which is impossible.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

362. Proposed by JAMES F. LAWRENCE, Stillwater, Okla.

Show that the number of solutions in positive integers, zero included, of the equation $x+2y+3z=6n$, is $3n^2+3n+1$.

Solution by W. J. GREENSTREET, M. A., Editor of the Mathematical Gazette, Burghfield, England.

$x+2y+3z=6n$. z may have any value from 0 to $2n$, inclusive.

Hence we may assign to it any even or odd value from 0 to $2n$, inclusive.